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NUMERICAL SOLUTION OF THE INVERSE PROBLEM OF HEAT CONDUCTION BY USING REGULARIZED DIFFERENCE SCHEMES

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The stability of difference schemes is investigated for the approximate solution of a multidimensional incorrect heat-conduction problem with inverse time.

Among the inverse problems of heat transfer [1], the problem with inverse time for the heat-conduction equation that belongs to the A. N. Tikhonov conditionally correct class attracts a great deal of attention. The general approach to the solution of unstable problems is formulated in [2] on the basis of the method of regularization. The method of quasiinversion [3] which consists in perturbing the initial equation has received wide propagation for differential equations. Of the later modifications of this method we note that described in [4] where a "pseudoparabolic" perturbation of the original equation as well as a "hyperbolic" modification are examined [1]. The stability of appropriate difference schemes of the quasi-inversion method is investigated in [5, 6].

Regularization of difference schemes is achieved in this paper by selecting a negative weight in the usual scheme with weights [7]. Economical difference schemes analogous to the locally one-dimensional schemes [7] in solving the direct heat conduction problem, are proposed in the multidimensional case. General results of the A. A. Samarskii [8] theory of stability of difference schemes are used in investigating the stability.

FORMULATION OF THE PROBLEM

Let Ω denote a n-dimensional parallelepiped: $\Omega = \{x|x = (x_1, x_2, \ldots, x_n), 0 < x_k < l_k, k = 1, 2, \ldots, n\}$.

For $x \in \Omega$ let us determine the uniform elliptical operator L:

$$Lu = \sum_{k=1}^{n} L_k u = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} a_k(x_k) \frac{\partial u}{\partial x_k}$$

with sufficiently smooth coefficients $a_k(x_k) \geqslant a_0 > 0,\ k = 1, 2, \ldots, n$. The function u(x, t) satisfies the heat-conduction equation with inverse time

$$\frac{\partial u}{\partial t} + Lu = 0,\ x \in \Omega,\ t \in S = (0, T),\ T > 0, \quad (1)$$

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and the boundary and initial conditions

$$u(x, t) = 0, \ x \in \partial\Omega, \ t \in S, \tag{2}$$

$$u(x, 0) = u_0(x), \ x \in \Omega. \tag{3}$$

In Ω we introduce a difference mesh ω_h that is uniform in every direction

$$\bar{\omega}_h = \omega_h + \gamma_h = \{x = (i_1 h_1, \ i_2 h_2, \ \ldots, \ i_n h_n),$$
$$i_k = 0, \ 1, \ \ldots, \ N_k, \ N_k h_k = l_k, \ k = 1, \ 2, \ \ldots, \ n\}.$$

Here ω_k is the set of inner, and γ_h the boundary nodes of the mesh. We approximate the operator L in ω_h by the mesh operator $\Lambda = \sum_{k=1}^{n} \Lambda_k$ of second order, where $\Lambda_k y = (d_k y_{\bar{x}_k})_{x_k}$ in the standard notation of the theory of difference schemes [7], while $d_k(x_k) = a_k(x_k - h_k/2)$. Taking account of the notation introduced, let us write a differential-difference equation for (1)-(3)

$$\frac{dy}{dt} + \Lambda y = 0, \ x \in \omega_h, \ t \in S, \tag{4}$$

with the additional conditions

$$y(x, t) = 0, \ x \in \gamma_h, \ t \in S, \tag{5}$$

$$y(x, 0) = u_0(x), \ x \in \omega_h. \tag{6}$$

Let us define the mesh operator A in the set of mesh functions that vanish on γ_h, where $Ay = \Lambda_y$ for $y = 0$, $x \in \gamma_h$. We write the problem (4)-(6) in the form of a first-order operator equation

$$\frac{dy}{dt} + Ay = 0, \ t \in S, \tag{7}$$

with the initial condition y^0. Let us note that the representation $A = \sum_{k=1}^{n} A_k$, is correct for A, where $A_k = A_k^* < 0$.

To construct the difference scheme for (7) we introduce a uniform mesh τ: $\omega_\tau = \{t = t_j = j\tau, \ j = 0, 1, \ldots, M, \ M\tau = T\}$. We will later keep the notation $y^j = y(t_j)$.

REGULARIZATION OF DIFFERENCE SCHEMES

We investigate the difference schemes for the problem formulated on the basis of the general theory of A. A. Samarskii [8]. Let us recall the fundamental result of the theory of stability of difference schemes that we need. The canonical form of a two-layered difference scheme has the form [7, 8]

$$B \frac{y^{j+1} - y^j}{\tau} + Ay^j = 0, \ j = 0, \ 1, \ \ldots, \ M - 1, \tag{8}$$

with given y^0. If the operators B and A in (8) are self-adjoint and constant (independent of j), then according to [8] a necessary and sufficient condition for the ρ-stability of the scheme (8) with any $\rho > 0$ is the bilateral operator inequality

$$\frac{1 - \rho}{\tau} B \leqslant A \leqslant \frac{1 + \rho}{\tau} B. \tag{9}$$

An appropriate estimate of the stability according to the initial data has the form

$$\|y^{j+1}\|_B = (By^{j+1}, \ y^{j+1})^{1/2} \leqslant \rho \|y^j\|_B,$$

where (., .) denotes the scalar product in the selected mesh space.

When considering unstable problems of the type (1)-(3), it is meaningful to speak about the ρ-stability of appropriate difference schemes with just $\rho = 1 + c\tau$, where c is a positive constant independent of the lattice spacings in space and time.

Let us consider an ordinary scheme with weights

$$\frac{y^{j+1} - y^j}{\tau} + A(\sigma y^{j+1} + (1-\sigma) y^j) = 0, \; j = 0, \; 1, \; \ldots, \; M-1 \tag{10}$$

for (7). It has the canonical form (8) for $B = E + \sigma\tau A$, where E is the unit mesh operator. Direct confirmation of conditions (9) for the explicit scheme ($\sigma = 0$) results in the fact that $\rho = 1 + c_0\tau$, where the constant c_0 is determined by the maximal eigenvalue of the operator A which has the order h^{-2} according to [7], where $h \sim h_1 \sim h_2 \sim \ldots \sim h_n$. Consequently, the mesh spacing in space will be the analog of the regularization parameter when applying the explicit scheme.

Significantly greater possibilities are provided when using schemes with negative weight, which we designate superexplicit. Let $\alpha = -\sigma\tau$ and $\alpha > 0$. For $\sigma < 0$ the operator B is positive definite and, consequently, the right hand inequality in (9) is satisfied. It is easy to see that the left side of (9) will be satisfied if we chose $\rho = 1 + \tau/\alpha$. Consequently, we obtain the necessary estimate for the stability of the difference solution

$$\|y^{j+1}\|_B \leqslant \left(1 + \frac{\tau}{\alpha}\right) \|y^j\|_B \leqslant \exp\left(\frac{\tau}{\alpha}\right) \|y^j\|_B \leqslant \exp\left(\frac{t_{j+1}}{\alpha}\right) \|y^0\|_B.$$

The quantity $\alpha = -\sigma\tau$ appears as regularization parameter when using the superexplicit schemes (10).

ECONOMICAL SCHEMES FOR MULTIDIMENSIONAL PROBLEMS

The economy of the difference scheme is of great value [7] for the numerical solution of multidimensional evolutionary problems. The superexplicit scheme (10) does not generally possess such a property since the mesh operator $B = E - \alpha A$ must be inverted on the upper layer. Taking account of the property of decomposability of the mesh operator A into pairwise commutatable self-adjoint operators A_k economic schemes are easily written down for (7). It is natural to consider a factorized scheme which is written in the canonical form (8) with the operator $B = \prod_{k=1}^{n} (E - \alpha A_k)$. It can be seen that such a scheme is also ρ-stable with $\rho = 1 + \tau/\alpha$. Let us note that it corresponds to application of a locally one-dimensional scheme for (7) of the form

$$\frac{y^{j+k/n} - y^{j+(k-1)/n}}{\tau} + A_k (\sigma_k y^{j+k/n} + (1-\sigma_k) y^{j+(k-1)/n}) = 0,$$

$$k = 1, \; 2, \; \ldots, \; n, \; j = 0, \; 1, \; \ldots, \; M-1,$$

for $\sigma_k = \sigma = -\alpha/\tau$.

The proposed approach was tested on a number of model problems analogous to those examined in [3]. Preliminary deepening of the investigation of the properties of the method in a narrower class of problems is necessary for its utilization in practical computations. This certainly also refers to other approximate methods of solving inverse heat-conduction problems.

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ANALYSIS OF THE ACCURACY OF SOLUTIONS OF THE TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEM

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UDC 536.2

The accuracy of solutions of the two-dimensional inverse heat-conduction problem is investigated. Exact and perturbed values of the temperature on the inner boundary are used as initial data.

Study of the nonstationary heating of structural elements, which are bodies of spherical and cylindrical shape and subjected to a high-temperature flux, requires knowledge of the external thermal loading conditions. The inverse problem, used to determine the conditions on the outer boundary according to the temperatures measured on the inner boundary, is examined below. In a number of cases it is necessary to take a two-dimensional heat-propagation model for bodies of spherical and cylindrical shape. For example, the two-dimensionality is taken into account for an intensive change in the free-stream flux parameters along the body generator and in the presence of anisotropy of the thermophysical properties [1]. A sufficiently large quantity of algorithms for solving inverse heat-transfer problems is known. Algorithms have been developed for solving inverse problems in linear and nonlinear formulations; algorithms taking into account structural changes in the material. These are mainly problems in a one-dimensional formulation which are justified in many cases of practical importance. For instance, if the installation of special heat-flux sensors is possible structurally in geometrically complicated spherical or cylindrical bodies, then in these cases there is no need to solve tedious multidimensional inverse problems. The determination of heat fluxes by using known heat-flux sensors is based on the solution of one-dimensional inverse heat-transfer problems. However, there exist few examples of practical investigations of the heat-transfer processes in constructions when the one-dimensional models do not adequately describe the actual physical processes and the installation of the above-mentioned heat-flux sensors is not possible. Spherical and cylindrical shells of small radius [2] are an example of such constructions. If there is a strict approach to the physical problem of heating, then a three-dimensional heat conductivity model is necessary to determine the external thermal boundary conditions for bodies of spherical and cylindrical shape. Unfortunately, a substantial growth of the calculations, resulting in large electronic computer time expenditure for the solution of the inverse problem, hinders the development of algorithms of inverse problems in a three-dimensional formulation. If the multidimensional nature of the heat conductivity in a cylindrical body is caused mainly by small radii of curvature, then the two-dimensional model describes the heat-conduction process well for a negligible heat-flux gradient along the cylinder generatrix. A two-dimensional heat-conduction model is also realized in the axisymmetric flow around a spherical body and the problem to determine the thermal boundary conditions can be solve in a polar coordinate system.

Let us consider a two-dimensional inverse boundary heat-conduction problem for a body of cylindrical shape. The heat flux $q_1(\varphi, \tau)$ is delivered to the outer surface, where τ is the time, φ is the angle of rotation in the cylindrical coordinate system 0, r, φ. As a result of the action of heat flux, a temperature field $T(r, \varphi, \tau)$ is realized in the body. We assume that the boundaries are heat insulated at $\varphi=0$, $\varphi=\varphi_k$ and $r = R_{ex}$. In this case the two-dimensional inverse heat-conduction problem is written as follows:

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